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whence

$$x = \frac{\phi(b)}{\phi(c)} \div \frac{\phi(a)}{\phi(c)} = \frac{\log_c b}{\log_c a},$$

and therefore

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

HARVARD UNIVERSITY,
JUNE, 1902.

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# ON POSITIVE QUADRATIC FORMS

BY PAUL SAUREL

THE necessary and sufficient conditions that a homogeneous quadratic function of $n$ variables be constantly positive or constantly negative are well known. A very simple demonstration of the necessity of these conditions has been given by Gibbs in his great memoir On the Equilibrium of Heterogeneous Substances.* This demonstration, however, has not received the attention which it deserves, perhaps because its simplicity is somewhat disguised by the physical terms employed. In the present note we shall reproduce Gibbs's demonstration and we shall complete it by showing that certain of the conditions thus obtained are sufficient.

Let us consider the quadratic function $\phi$ defined by the equation

$$\phi = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k, \tag{1}$$

in which

$$a_{ik} = a_{ki}, \tag{2}$$

and let us write

$$f_i = \sum_{k=1}^{n} a_{ik} x_k. \tag{3}$$

From (3) we get

$$df_i = \sum_{k=1}^{n} a_{ik} dx_k, \tag{4}$$

* *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 3, part 1, page 166 (1876).

and from this in turn we get

$$\sum_{i=1}^{n} df_i dx_i = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} dx_i dx_k. \tag{5}$$

From equations (1) and (5) it follows, when we remember that the differentials of the independent variables are entirely arbitrary, that the necessary and sufficient condition that $\phi$ be always positive is that

$$\sum_{i=1}^{n} df_i dx_i > 0. \tag{6}$$

By giving to the differentials in this inequality special values, we can deduce from it a great variety of necessary conditions. Certain of these, which we shall ultimately prove to be sufficient conditions also, we now deduce.

Setting $dx_1 \neq 0$, $dx_2 = dx_3 = \cdot \cdot \cdot \cdot = dx_n = 0$, and dividing (6) by the positive quantity $dx_1^2$, we have as a first necessary condition

$$\frac{\partial f_1}{\partial x_1} > 0.$$

Let us now introduce in place of the independent variables $x_1, x_2, \cdot \cdot \cdot, x_n$ the new independent variables $f_1, x_2, \ldots, x_n$. This will be possible if $\partial f_1/\partial x_1 \neq 0$, and therefore, in particular, if the necessary condition just obtained is fulfilled. Now set $dx_2 \neq 0$, $df_1 = dx_3 = \cdot \cdot \cdot \cdot \cdot = dx_n = 0$ and divide (6) by $dx_2^2$ getting as a second necessary condition

$$\frac{\partial f_2}{\partial x_2} > 0,$$

where, however, we must remember in forming the partial derivative that the independent variables are now $f_1, x_2, \ldots, x_n$.

Next pass from the independent variables just used to the independent variables $f_1, f_2, x_3, \ldots, x_n$ — a change of variable which can be made if the last written inequality is fulfilled — and proceed as before. In this way we get the following set of necessary conditions:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, x_3, \ldots, x_n} > 0,$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, x_3, \ldots, x_n} > 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \tag{7}$$

$$\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)_{f_1, \ldots, f_{n-2}, x_{n-1}, x_n} > 0,$$

$$\left(\frac{\partial f_n}{\partial x_n}\right)_{f_1, \ldots, f_{n-2}, f_{n-1}, x_n} > 0,$$

in which the subscripts indicate the quantities which are regarded as the independent variables.

The quantities which appear in inequalities (7) can be put into another form. Denote by $\Delta_n$ the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

and by $\Delta_{n-1}, \Delta_{n-2}, \ldots, \Delta_1$ the determinants obtained by erasing the last row and the last column of $\Delta_n, \Delta_{n-1}, \ldots, \Delta_2$. If we make use of equations (3), we obtain without difficulty the following equations:

$$\left(\frac{\partial f_1}{\partial x_1}\right)_{x_1, x_2, x_3, \ldots, x_n} = \Delta_1,$$

$$\left(\frac{\partial f_2}{\partial x_2}\right)_{f_1, x_2, x_3, \ldots, x_n} = \frac{\Delta_2}{\Delta_1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \tag{8}$$

$$\left(\frac{\partial f_{n-1}}{\partial x_{n-1}}\right)_{f_1, \ldots, f_{n-2}, x_{n-1}, x_n} = \frac{\Delta_{n-1}}{\Delta_{n-2}},$$

$$\left(\frac{\partial f_n}{\partial x_n}\right)_{f_1, \ldots, f_{n-2}, f_{n-1}, x_n} = \frac{\Delta_n}{\Delta_{n-1}}.$$

Conditions (7) may accordingly be replaced by the conditions:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \ldots, \quad \Delta_n > 0. \tag{9}$$

The demonstration by which we have established the necessity of these conditions is that of Gibbs.

We shall now show that conditions (7) are not only necessary but are also sufficient. For this purpose it will be enough to show that if conditions of the form (7) are sufficient when $n - 1$ variables are involved, they are also sufficient when $n$ variables are involved.

By referring to equations (1) and (3) it is obvious that we can write

$$\phi = \sum_{i=1}^{n} f_i x_i. \tag{10}$$

We can throw this equation into the form

$$\phi = \frac{f_1^2}{a_{11}} + \sum_{i=2}^{n} f_i' x_i, \tag{11}$$

where

$$f_i' = \frac{a_{11} f_i - a_{1i} f_1}{a_{11}}. \tag{12}$$

It should be noticed that $f_i'$ is independent of $x_1$.

If conditions (7) are sufficient when $n - 1$ variables are involved it follows from equation (11) that $\phi$ will certainly be positive if the following conditions hold:

$$\begin{gathered} a_{11} > 0, \\ \left(\frac{\partial f_2'}{\partial x_2}\right)_{x_2, x_3, x_4, \ldots, x_n} > 0, \\ \left(\frac{\partial f_3'}{\partial x_3}\right)_{f'_2, x_3, x_4, \ldots, x_n} > 0, \\ \cdots\cdots\cdots \\ \left(\frac{\partial f_n'}{\partial x_n}\right)_{f'_2, f'_3, \ldots, f'_{n-1}, x_n} > 0. \end{gathered} \tag{13}$$

Since $f_i'$ is independent of $x_1$, we may give to $dx_1$ any convenient value. It will therefore be allowable to suppose that in each of the differential coefficients in (13) $dx_1$ has been so taken that

$$df_1 = 0. \tag{14}$$

But, in that case, equation (12) shows that

$$df_i' = df_i. \tag{15}$$

If we make use of equations (14) and (15), conditions (13) reduce at once to conditions (7).

Thus, if conditions (7) be sufficient conditions in the case of $n-1$ variables they are also sufficient conditions in the case of $n$ variables. As these conditions are obviously sufficient in the case of one variable, they are sufficient in general.

If we wish to obtain the necessary and sufficient conditions that $\phi$ be constantly negative we must reverse the sign of inequality in each of conditions (7). The signs of the determinants in (9) will then be alternately negative and positive.

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